Period tripling accumulation point for complexified Hénon map

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Abstract

Accumulation point of period-tripling bifurcations for complexified Hénon map is found. Universal scaling properties of parameter space and Fourier spectrum intrinsic to this critical point is demonstrated.

It is known that complexification of real 1D logistic map

$$z_{n+1} = \lambda - z_n^2, \tag{1}$$

where $\lambda, z \in \mathbb{C}$ leads to the origination of the Mandelbrot set at the complex parameter λ plane [1] and a number of other accompanying phenomena. So, except for the transition to chaos through cascade of period-doubling bifurcations, intrinsic to the real logistic map, other bifurcation sequences are possible. For example, the universal properties of transition to chaos through period-tripling cascade is studied in paper of Golberg-Sinai-Khanin [2]. Corresponded critical point (GSK point) is situated at

$$\lambda_c = 0.0236411685377 + 0.7836606508052i \tag{2}$$

and characterized by following critical indexes [2], namely, by critical multiplier μ_c, scale factor α and parameter scaling constant δ (see Fig.1(a)):

$$\mu_c = -0.47653179 - 1.05480867i, \tag{3}$$

$$\alpha = -2.0969 + 2.3583i, \tag{4}$$

$$\delta = 4.6002 - 8.9812i. \tag{5}$$

Opportunity for realization of the phenomena, characteristic for the dynamics of complex maps (Mandelbrot set etc.) at the physical systems seems to be interesting problem [3]. In the context of this problem the following question is important: Does phenomena of dynamics of the 1D complex maps like classic Mandelbrot map (1) survive for the two-dimensional maps. For example, from the point of view of possible physical applications, more realistic model rather than logistic map, is the Hénon map

$$z_{n+1} = f(z_n, w_n) = \lambda - z_n^2 - d \cdot w_n, \qquad w_{n+1} = g(z_n, w_n) = z_n. \tag{6}$$

In the real variable case the system (6) is 2D invertible map and, hence, can be realized as Poincare cross-section of flow system with three dimensional phase space – minimal dimension, providing opportunity of nontrivial dynamics and chaos. Hénon map is suitable for modelling of the chaotic dynamics of the generator with non-inertial nonlinearity, dissipative oscillator and rotator with periodic impulse driving force etc. [4]. Moreover, Hénon map expresses the principal properties of large class of differential systems.

Let us complexify the map (6) in a such way that $z, w, \lambda \in \mathbb{C}$, $d \in \mathbb{R}$. According to the work [5, 6], complexified Hénon map can be reduced to the two symmetrically coupled real Hénon maps and can be realised at the physical experiment [7].

Let us remark that with $|d| < 1$ the Hénon map is dissipative system, with $|d| \to 1$ – it is area-preserving map, and with $d \to 0$ it corresponds to complex 1D quadratic map (1), which describes the universal scenario of transition to chaos through period-tripling bifurcations. In present work we aim to be convinced of existence of GSK critical point for the Hénon map with $1 > |d| \neq 0$, to find this point and to demonstrate intrinsic to it scaling characteristics.

Let us shortly explain essence of the numerical procedure for the calculation of the GSK point. From the work [8], the simple and effective method of calculation of the critical point of transition to chaos though period-doubling bifurcations for 1D maps and also for the more general systems is known. We have generalized this method to the case of GSK critical point and have applied it to the 2D complex Hénon map. The method is based on the fact, that at the GSK point the infinite number of the unstable cycles with tripling periods must exist, and multipliers of these cycles must tend to the universal constant μ_c. Therefore, for the finding of the point $\lambda_c^{d\neq 0}$ it is enough to calculate with fixed parameter d the point at the λ plane, satisfying to these conditions. Therefor it is necessary to solve the system of nonlinear complex equations:

$$f^N(z_N, w_N) = z_N, \quad g^N(z_N, w_N) = w_N, \quad \mu(z_N, w_N) = \mu_c \tag{7}$$

concerning the cycle elements z_N, w_N, where $N = 3^k$ ($k \to \infty$) and parameter λ. The system of equations (7) can be solved numerically by Newton procedure. The multiplier $\mu(z_N, w_N)$ can be found as maximal by absolute value eigenvalue of the Jacobi matrix $\mathbf{J}(z_N, w_N)$, where

$$\mathbf{J}(z,w) = \begin{pmatrix} \frac{\partial f^N(z,w)}{\partial z} & \frac{\partial g^N(z,w)}{\partial z} \\ \frac{\partial f^N(z,w)}{\partial w} & \frac{\partial g^N(z,w)}{\partial w} \end{pmatrix}. \tag{8}$$

Let us mark, that for the best convergence of a Newton method, expediently to use inheriting starting conditions. At the each consequent step of calculations, as a starting position of the critical point the value of the parameter λ obtained at the previous step is used. As starting position of the elements of cycles the values obtained according to the universal scaling properties of the phase space at the critical GSK point are used. First elements of different cycles selected properly should obey the equations

$$z_N = z_c + C_1\alpha^{-N}, \qquad w_N = w_c + C_2\alpha^{-N}, \tag{9}$$

where z_c and w_c – are the constants, which correspond to the scaling center at the phase space, and C_1, C_2 –are some complex coefficients. In the case of the 1D complex map, i.e. with $d = 0$, the scaling center is situated at the origin. With $d \neq 0$ the scaling center appears displaced concerning an origin of coordinates and can be found from the following conditions

$$z_c = (\alpha z_{N+1} - z_N)/(\alpha - 1), \qquad w_c = (\alpha w_{N+1} - w_N)/(\alpha - 1). \tag{10}$$

Let us summarize the algorithm of numerical calculations. At the first stage, one must define the critical GSK point for the cycles with not very large periods and dragging the parameter d from 0 (critical point of the 1D map) to the necessary value. This allow to use the known critical GSK point (2) as a starting condition at the beginning of the calculation procedure at $d = 0$. Then, with fixed parameter d one must consequently improve the critical point using the more long-periodic cycles.

Numerical calculations have shown that GSK point for the Hénon map (6) with traditional, mentioned in the original work of Hénon [9] value of parameter $d = -0.3$ is situated at

$$\lambda_c^{d=-0.3} = -0.24388583757 + 0.69478896727i. \tag{11}$$

The position of the scaling center is defined by the following values

$$z_c = -0.05289397 + 0.15014466i, \qquad w_c = 0.56676843 + 0.72274977i. \tag{12}$$

In the neighborhood of the point (11) at the parameter plane (Reλ, Imλ) the scaling, characterized by constant δ takes place (see Fig. 1(b)). The existence of the infinite number of the cycles with

tripling periods immediately at the critical point $\lambda_c^{d=-0.3}$ is proved to be true by the structure of the Fourier spectrum. At Figure 2 the dependence of the spectral intensities versus the logarithm of frequency f is represented, and amplitudes of harmonics are rated to the f^{κ}, where $\kappa = 6.15$ – is universal constant, characterizing the self-similar fractal structure of the spectrum at the GSK point [10].

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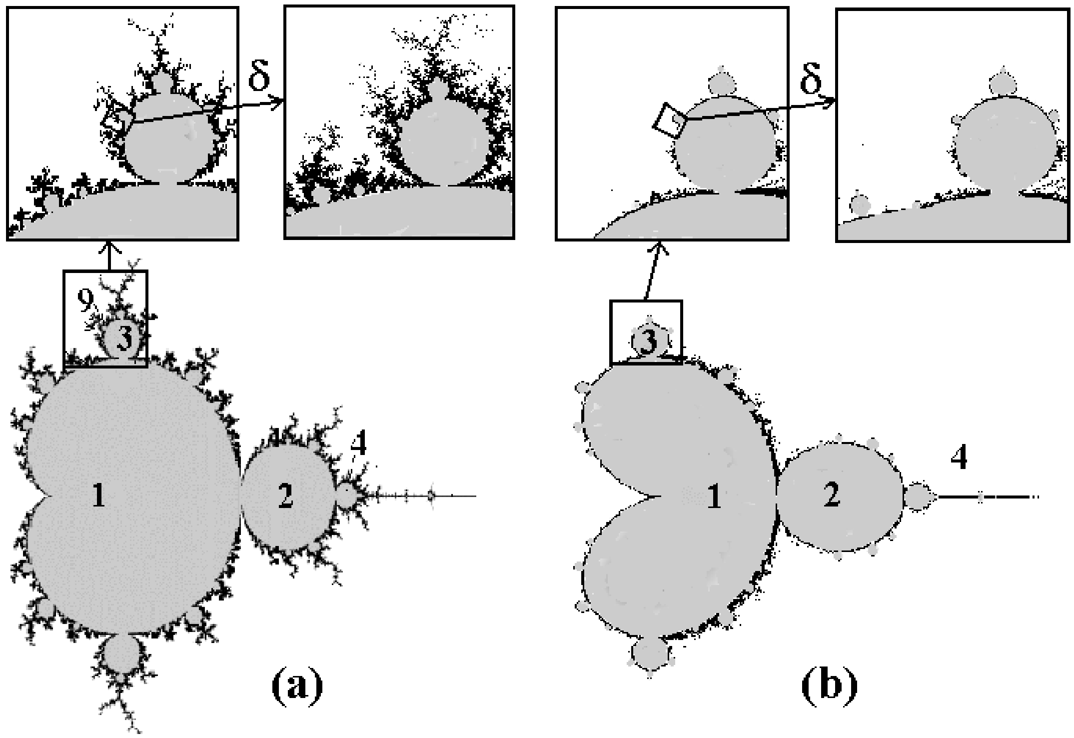

Figure 1: Demonstration of the scaling properties of the Mandelbrot set near the GSK point (critical point is situated at the center of the small fragments) for the complexified Hénon map (5) at the plane of complex parameter λ with $d = -0.0$ (a) and $d = -0.3$ (b). Performed by the multiplication to the complex scaling constant δ the fragment of the Mandelbrot set has the same structure with the previous fragment.

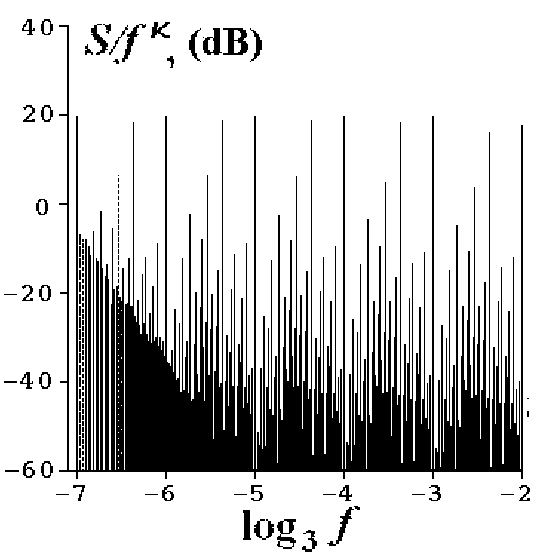

Figure 2: Double logarithmic plot of Fourier spectrum of the signal, originated by the map (6) at the critical point (11). Harmonics of the tripling frequencies has the same amplitudes.